CHAPTER 15

Imaginary Numbers

LEARNING OBJECTIVE

After completing this chapter, you will be able to:

- Perform arithmetic operations on imaginary and complex numbers

5/600 SmartPoints® (Low Yield)

How Much Do You Know?

Directions: Try the questions that follow. Show your work so that you can compare your solutions to the ones found in the Check Your Work section immediately after this question set. If you answered most of the questions correctly, and if your scratchwork looks like ours, you may be able to move quickly through this chapter. If you answered incorrectly or used a different approach, you may want to take your time on this chapter.

1. Which of the following is the correct simplification of the expression $(2i - 3) - (6 + 4i)$, where $i = \sqrt{-1}$?

A) $-9 - 2i$

B) $-9 + 6i$

C) $-7 - 4i$

D) $3 + 6i$

2. Given that $i = \sqrt{-1}$, which of the following is equivalent to $(6i^2 - 7i) + (3 + 6i)$?

A) $6i^2 + i + 3$

B) $-3 - i$

C) $-9 - i$

D) 10

3. Which of the following is equal to $(17 + 7i)(3 - 5i)$? (Note: $i = \sqrt{-1}$)

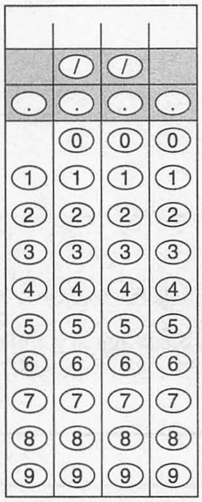
A) 16

B) 86

C) $16 - 64i$

D) $86 - 64i$

4. If the expression $\frac{2-i}{2+i}$ is written as the complex number $a + bi$, where a and b are real numbers, what is the value of a ? (Note: $i = \sqrt{-1}$)



Answers and explanations are on the next page. ▶ ▶ ▶

Check Your Work

1. A

Difficulty: Easy

Getting to the Answer: The real and imaginary components of the two binomials aren't in the same order. Distributing the negative sign inside the second set of parentheses produces $2i - 3 - 6 - 4i$. Combine like terms to obtain $-9 - 2i$, which makes **(A)** correct.

2. B

Difficulty: Easy

Getting to the Answer: Treat i like a variable and simplify the expression by getting rid of the parentheses and combining like terms:

$$\left(6i^2 - 7i\right) + (3 + 6i)$$
$$= 6i^2 - 7i + 3 + 6i$$
$$= 6i^2 - i + 3$$

Notice that the resulting expression almost, but not quite, matches (A), so eliminate (A). Because $i = \sqrt{-1}$, you can square both sides to find that $i^2 = -1$. Plug in -1 for i^2 and combine like terms:

$$6(-1) - i + 3$$
$$= -i - 3$$

Rewrite in $a + bi$ form: $-3 - i$. **(B)** is correct.

3. D

Difficulty: Medium

Getting to the Answer: Use FOIL to multiply the binomials and simplify:

$$(17 + 7i)(3 - 5i)$$
$$= (17 \times 3) + \left(17 \times (-5i)\right) + (7i \times 3) + \left(7i \times (-5i)\right)$$
$$= 51 - 85i + 21i - 35i^2$$
$$= 51 - 64i - \left(35 \times (-1)\right)$$
$$= 86 - 64i$$

Choice **(D)** is correct.

4. 3/5

Difficulty: Medium

Getting to the Answer: The question is really asking you to manipulate this expression out of its fraction form to match the form $a + bi$ and identify the real component a. To rationalize the denominator, use the conjugate of the denominator, $2 - i$, to multiply the expression by 1:

$$\left(\frac{2-i}{2+i}\right)\left(\frac{2-i}{2-i}\right)$$
$$= \frac{4 - 2i - 2i + i^2}{4 + 2i - 2i - i^2}$$

Since squaring $\sqrt{-1}$ would result in -1, plug in -1 for i^2 and combine like terms:

$$\frac{4 - 2i - 2i + (-1)}{4 + 2i - 2i - (-1)}$$
$$= \frac{3 - 4i}{5}$$

Splitting the terms so that the expression matches the form $a + bi$ results in $\frac{3}{5} - \frac{4}{5}i$. Grid in **3/5**.

Arithmetic Operations with Complex Numbers

LEARNING OBJECTIVE

After this lesson, you will be able to:

- Perform arithmetic operations on imaginary and complex numbers

To answer a question like this:

Which of the following complex numbers is equivalent to $\frac{2+i}{3+5i}$?

A) $\frac{2}{3}+\frac{i}{5}$

B) $\frac{2}{3}-\frac{i}{5}$

C) $\frac{11}{34}+\frac{7i}{34}$

D) $\frac{11}{34}-\frac{7i}{34}$

You need to know this:

The square root of a negative number is not a real number but an **imaginary number**.

To take the square root of a negative number, use ***i***, which is defined as $i=\sqrt{-1}$. For example, to simplify $\sqrt{-49}$, rewrite $\sqrt{-49}$ as $\sqrt{-1\times 49}$, take the square root of -1 (which is by definition *i*), and then take the square root of 49, which is 7. The end result is $7i$.

The simplification $i^2=\left(\sqrt{-1}\right)^2=-1$ is also useful when working with imaginary numbers. For example:

$$\begin{aligned}&\sqrt{-16}\times\sqrt{-25}\\&=i\sqrt{16}\times i\sqrt{25}\\&=4i\times 5i\\&=20i^2\\&=20\times(-1)\\&=-20\end{aligned}$$

When a number is written in the form $a+bi$, where a is the real component and b is the imaginary component (and i is $\sqrt{-1}$), it is referred to as a **complex number**.

You need to do this:

You can add, subtract, multiply, and divide complex numbers just as you do real numbers:

- To add (or subtract) complex numbers, simply add (or subtract) the real parts and then add (or subtract) the imaginary parts.
- To multiply complex numbers, treat them as binomials and use FOIL. To simplify the product, use the simplification $i^2 = -1$ and combine like terms.
- To divide complex numbers, write them in fraction form and multiply the numerator and denominator by the **conjugate** of the complex number in the denominator. To form the conjugate, change the sign in the complex number. For example, the conjugate of $2 + i$ is $2 - i$.

Explanation:

Use the conjugate to simplify complex numbers in the denominator. The conjugate of $3 + 5i$ is $3 - 5i$, so multiply the expression by 1 in the form $\frac{3-5i}{3-5i}$ and simplify:

$$\frac{2+i}{3+5i} \times \frac{3-5i}{3-5i} = \frac{6-10i+3i-5i^2}{(3+5i)(3-5i)} = \frac{6-7i-5i^2}{9-25i^2}$$

You know $i^2 = -1$, so the expression simplifies to $\frac{6-7i-5(-1)}{9-25(-1)} = \frac{11-7i}{34}$. To separate the complex expression into its real and imaginary components, write each of the terms in the numerator over the denominator in separate fractions, which produces $\frac{11}{34} - \frac{7i}{34}$. Choice **(D)** is correct.

Try on Your Own

Directions: Take as much time as you need on these questions. Work carefully and methodically. There will be an opportunity for timed practice at the end of the chapter.

HINT: For Q1, what is the value of i^2 ?

1. Given that $i = \sqrt{-1}$, what is the sum of the complex numbers $(21i^2 - 12i) + (3 - 5i)$?

A) 7

B) $7i$

C) $-18 - 17i$

D) $24 - 17i$

2. What is the result of the multiplication $(5 - 6i)(3 + 3i)$? (Note: $i = \sqrt{-1}$)

A) $33 - 3i$

B) $23 - 3i$

C) $11 - i$

D) -6

HINT: For Q3, multiply the top and bottom of the fraction by the conjugate of the denominator.

3. Which of the following is equivalent to $\frac{38 + 18i}{4 + 6i}$? (Note: $i = \sqrt{-1}$)

A) $\frac{11}{13} - 3i$

B) $5 - 3i$

C) $\frac{19}{2} + 3i$

D) $44 + 300i$

$$\frac{7 + i}{8 - i}$$

4. If the expression above is expressed in the form $a + bi$, where $i = \sqrt{-1}$, what is the value of b ?

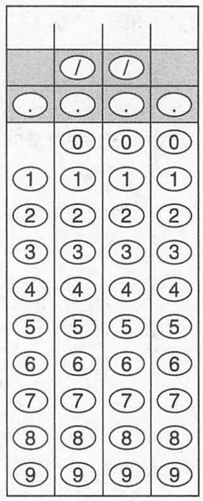
A) -1

B) $\frac{3}{13}$

C) $\frac{11}{13}$

D) $\frac{15}{64}$

5. If the expression $\frac{-3i^2 + 2i}{2 + i}$ is written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$, what is the value of a ?



Math

On Test Day

The SAT tests the same patterns over and over again, but they can be presented in a number of different ways. Before you start doing calculations, look closely at this question and see if you recognize a pattern that will help you simplify it. Once you spot the pattern, this question should take you no more than a few seconds to solve.

$$\frac{25i^2 - 9}{5i + 3}$$

6. For $i = \sqrt{-1}$, the expression above is equal to which of the following complex numbers?

A) $5i - 3$

B) $5i + 3$

C) $16i - 9$

D) $80i - 48$

The correct answer and explanation can be found at the end of the chapter.

How Much Have You Learned?

Directions: For testlike practice, give yourself 7 minutes to complete this question set. Be sure to study the explanations, even for questions you got right. They can be found at the end of this chapter.

$$(3 + 4i) - (2 + 3i)$$

7. Given that $i = \sqrt{-1}$, what is the value of the expression above?

A) $1 - i$

B) $1 + i$

C) $1 + 7i$

D) $5 + 7i$

$$\frac{3i + 2}{-i - 3}$$

8. If the expression above is expressed in the form $a + bi$, where $i = \sqrt{-1}$, what is the value of b ?

A) -0.7

B) 0.7

C) -0.9

D) 0.9

$$\frac{5 + 3i}{7 - 3i}$$

9. Which of the following expressions is equivalent to the expression above, assuming that $i = \sqrt{-1}$?

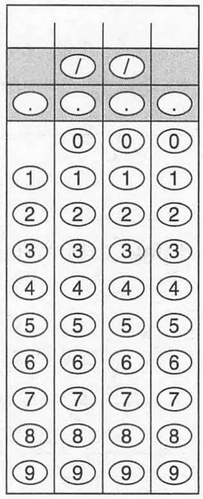
A) $\frac{1}{5}$

B) $\frac{5}{7}$

C) $\frac{13 + 18i}{20}$

D) $\frac{13 + 18i}{29}$

10. If the expression $\frac{1 + 2i}{4 + 2i}$ is rewritten as a complex number in the form of $a + bi$, what is the value of a ? (Note: $i = \sqrt{-1}$)



11. Given that $i = \sqrt{-1}$, what is the product of $\frac{2 - 4i + 2i^2}{2}$ and $\frac{1}{1 - i}$?

A) $-1 + i$

B) $-1 - i$

C) $1 - i$

D) $1 + i$

Math

Reflect

Directions: Take a few minutes to recall what you've learned and what you've been practicing in this chapter. Consider the following questions, jot down your best answer for each one, and then compare your reflections to the expert responses on the following page. Use your level of confidence to determine what to do next.

In a complex number in the form $a + bi$, what is the definition of i?

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What is the procedure for adding, subtracting, or multiplying complex numbers?

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What is the procedure for dividing complex numbers?

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Expert Responses

In a complex number in the form $a + bi$, what is the definition of i?

In any complex number written as $a + bi$, $i = \sqrt{-1}$.

What is the procedure for adding, subtracting, or multiplying complex numbers?

To perform arithmetic operations on complex numbers, treat i *as a variable and simplify* i^2 *to* -1.

What is the procedure for dividing complex numbers?

To divide two complex numbers, write the division as a fraction, multiply the numerator and denominator by the conjugate of the denominator, and simplify.

Next Steps

If you answered most questions correctly in the "How Much Have You Learned?" section, and if your responses to the Reflect questions were similar to those of the SAT expert, then consider imaginary numbers an area of strength and move on to the next chapter. Come back to this topic periodically to prevent yourself from getting rusty.

If you don't yet feel confident, review those parts of this chapter that you have not yet mastered, then try the questions you missed again. As always, be sure to review the explanations closely.

Math

Answers and Explanations

1. C

Difficulty: Easy

Getting to the Answer: Plug in -1 for i^2 and combine like terms. Remember to treat i like a variable:

$$\left(21i^2 - 12i\right) + (3 - 5i)$$
$$= 21(-1) - 12i + 3 - 5i$$
$$= -18 - 17i$$

(C) is correct.

2. A

Difficulty: Medium

Getting to the Answer: Treat i as a variable and expand the expression as if it were a binomial. Use FOIL to distribute and then plug in -1 for any i^2 and combine like terms:

$$(5 - 6i)(3 + 3i)$$
$$= 15 + 15i - 18i - 18i^2$$
$$= 15 - 3i - 18(-1)$$
$$= 33 - 3i$$

(A) is correct.

3. B

Difficulty: Hard

Getting to the Answer: Multiply the numerator and denominator by the latter's conjugate and then use FOIL. Combine like terms and reduce fractions as needed:

$$\frac{38 + 18i}{4 + 6i}$$
$$= \frac{38 + 18i}{4 + 6i} \times \frac{4 - 6i}{4 - 6i}$$
$$= \frac{(38 \times 4) + (38 \times -6i) + (18i \times 4) + (18i \times -6i)}{16 - 36i^2}$$
$$= \frac{152 - 228i + 72i - 108i^2}{16 - (-36)}$$
$$= \frac{152 - 156i - (-108)}{52}$$
$$= \frac{260 - 156i}{52}$$
$$= \frac{260}{52} - \frac{156}{52}i$$
$$= 5 - 3i$$

(B) is correct.

4. B

Difficulty: Medium

Getting to the Answer: The question asks for the value of b when a certain complex number is in the form $a + bi$. The given expression has a complex number in the denominator, so start by multiplying the top and bottom by the conjugate of the complex number in the denominator:

$$\left(\frac{7 + i}{8 - i}\right)\left(\frac{8 + i}{8 + i}\right)$$
$$= \frac{56 + 7i + 8i + i^2}{64 + 8i - 8i - i^2}$$

Since squaring $\sqrt{-1}$ results in -1, plug in -1 for i^2 and then combine like terms and reduce.

$$\frac{56 + 7i + 8i + (-1)}{64 + 8i - 8i - (-1)}$$
$$= \frac{55 + 15i}{65}$$
$$= \frac{\not{5}(11 + 3i)}{\not{5}(13)} = \frac{11 + 3i}{13}$$

Next, write each of the terms in the numerator over the denominator in separate fractions so that the expression matches the form $a + bi$: $\frac{11}{13} + \frac{3i}{13}$. Thus, $b = \frac{3}{13}$, so **(B)** is correct.

5. 8/5 or 1.6

Difficulty: Medium

Getting to the Answer: First, notice that the expression simplifies if you plug in -1 for i^2 in the numerator. Next, rationalize the denominator by multiplying the top and bottom by the conjugate of the denominator:

$$\left(\frac{-3(-1) + 2i}{2 + i}\right)\left(\frac{2 - i}{2 - i}\right)$$
$$= \frac{6 + 4i - 3i - 2i^2}{4 + 2i - 2i - i^2}$$

Notice again that plugging in -1 for i^2 and combining like terms results in more simplification:

$$\frac{6+4i-3i-2(-1)}{4+2i-2i-(-1)}$$
$$=\frac{8+i}{5}$$
$$=\frac{8}{5}+\frac{i}{5}$$

Since the question asks for the value of a, grid in **8/5** or **1.6**.

6. A

Difficulty: Medium

Strategic Advice: The expression in the numerator contains a coefficient and a constant that are both perfect squares, with the constant being subtracted. This pattern signals that the numerator is a difference of perfect squares, which can be factored.

Getting to the Answer: While you could rationalize the given fraction with the denominator's conjugate, in this case, the difference of perfect squares in the numerator means you can factor it into binomials and then cancel a factor from the numerator and denominator:

$$\frac{25i^2-9}{5i+3}$$
$$=\frac{(5i+3)(5i-3)}{5i+3}$$
$$=\frac{\cancel{(5i+3)}(5i-3)}{\cancel{5i+3}}$$
$$=5i-3$$

Thus, **(A)** is correct.

7. B

Difficulty: Easy

Getting to the Answer: To subtract complex numbers, simply subtract the real parts and then subtract the imaginary parts. Remember to distribute the -1 to both 2 and $3i$ in the second binomial:

$$(3+4i)-(2+3i)$$
$$=3+4i-2-3i$$
$$=1+i$$

(B) is correct.

8. A

Difficulty: Medium

Getting to the Answer: Rewrite the expression so that each complex number is in the standard $a+bi$ form. Then rationalize using the conjugate of the denominator:

$$\left(\frac{3i+2}{-i-3}\right)\Rightarrow\left(\frac{2+3i}{-3-i}\right)$$
$$=\left(\frac{2+3i}{-3-i}\right)\times\left(\frac{-3+i}{-3+i}\right)$$
$$=\frac{-6-9i+2i+3i^2}{9+3i-3i-i^2}$$

Combine like terms and plug in -1 for i^2:

$$\frac{-6-7i+3i^2}{9-i^2}$$
$$=\frac{-6-7i+3(-1)}{9-(-1)}$$
$$=\frac{-9-7i}{10}$$
$$=-\frac{9}{10}-\frac{7i}{10}$$

Since the question asks for the value of b, convert the fraction to a decimal. **(A)** is correct.

9. D

Difficulty: Medium

Strategic Advice: If part of an answer choice is incorrect, then the whole answer choice is incorrect. Notice that multiplying the numerator by the conjugate of the denominator will result in a complex number in the numerator, so eliminate (A) and (B).

Getting to the Answer: Multiply the expression by the conjugate of the denominator to remove the complex number from the denominator. Note that the two remaining answer choices have the same numerator, $13 + 18i$, which means that you need to simplify the denominator only to decide which is correct. For the record, the simplification of the numerator is included:

$$\frac{5+3i}{7-3i} \times \frac{7+3i}{7+3i}$$

$$= \frac{35+21i+15i+9i^2}{49-21i+21i-9i^2}$$

$$= \frac{35+36i+9i^2}{49-9i^2}$$

Next, plug in -1 for i^2 and combine like terms. The resulting fraction can be reduced by factoring out 2:

$$\frac{35+36i+9(-1)}{49-9(-1)}$$

$$= \frac{26+36i}{58}$$

$$= \frac{\cancel{2}(13+18i)}{\cancel{2}(29)}$$

$$= \frac{13+18i}{29}$$

(D) is correct.

10. 8/20 or 4/10 or 2/5 or .4

Difficulty: Medium

Getting to the Answer: The question wants the expression to be in the form $a + bi$. When there is a complex number in the denominator, multiply the top and bottom of the fraction by the conjugate of the the denominator. Plug in -1 for i^2 and combine like terms:

$$\frac{1+2i}{4+2i} \times \frac{4-2i}{4-2i}$$

$$= \frac{4+8i-2i-4i^2}{16-8i+8i-4i^2}$$

$$= \frac{4+6i-4(-1)}{16-4(-1)}$$

$$= \frac{8+6i}{20}$$

$$= \frac{8}{20} + \frac{6i}{20}$$

Grid in **8/20, 4/10, 2/5,** or **.4**. Any of these formats is correct.

11. C

Difficulty: Hard

Strategic Advice: If you factor out the 2, the expression in the numerator of the first fraction is a classic quadratic $x^2 - 2xy + y^2 = (x - y)^2$. Factor the quadratic and and look for terms that cancel.

Getting to the Answer: Multiply the two fractions and factor out a 2. After factoring the classic quadratic, the term $(1 - i)$ cancels out:

$$\left(\frac{2-4i+2i^2}{2}\right)\left(\frac{1}{1-i}\right)$$

$$= \frac{2(1-2i+i^2)}{2(1-i)}$$

$$= \frac{2(1-i)^2}{2(1-i)}$$

$$= \frac{\cancel{2}\cancel{(1-i)}(1-i)}{\cancel{2}\cancel{(1-i)}}$$

$$= 1 - i$$

(C) is correct.