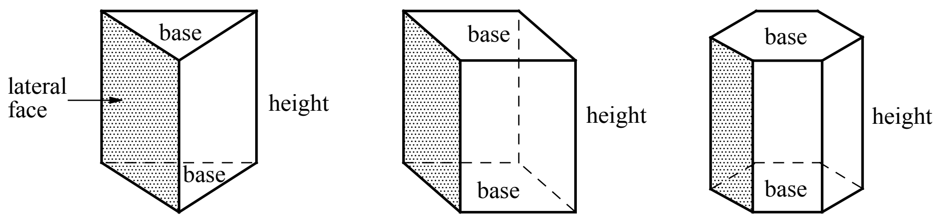
CHAPTER 20
Surface Areas and Volumes

20-1. Prisms

A **prism** is a polyhedron (a space figure whose faces are polygons) with two congruent and parallel faces. These two faces are called the **bases** of the prism. The other faces are **lateral faces**. In a prism, the lateral faces are rectangles. The **height** of a prism is a segment joining the two base planes and is perpendicular to both.



Triangular prism   Rectangular prism   Hexagonal prism

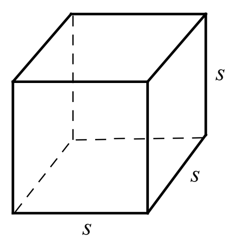
Theorems - Prisms

The **lateral area** of a prism is the perimeter of a base times the height of the prism. $L.A. = P \cdot h$

The total **surface area** of a prism is the sum of the lateral area and the areas of the two bases. $T.A. = L.A. + 2B$

The **volume** of a prism is the area of a base times the height of the prism. $V = B \cdot h$

A **cube** is a prism in which all the faces are squares.



Theorems - Cubes

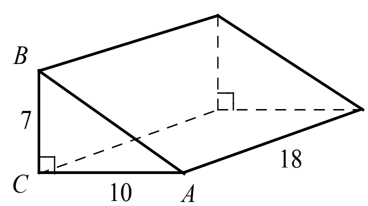
Volume of a cube $= s^3$

Surface area of a cube $= 6s^2$

Example 1 □ A triangular prism is shown at the right.

a. Find the area of the base.

b. Find the volume of the prism.



Solution □ a. $B = \frac{1}{2}(7)(10) = 35$ $B =$ area of triangle ABC.

b. $V = 35 \cdot 18 = 630$ $V = B \cdot h$

Example 2 □ A cube has a volume of 1,728 cubic inches (in^3). What is the surface area of the cube, in square inches?

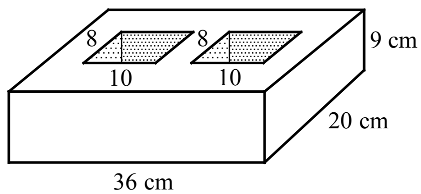
Solution □ Let $s =$ the length of the cube.

$$V = s^3 = 1,728 \text{ in}^3 \Rightarrow s = \sqrt[3]{1728 \text{ in}^3} = (1728 \text{ in}^3)^{\frac{1}{3}} = 12 \text{ in}$$

Surface area of a cube $= 6s^2 = 6(12 \text{ in})^2 = 864 \text{ in}^2$

Exercises – Prisms

1



The figure above shows a cement block of $36\text{ cm}\times 20\text{ cm}\times 9\text{ cm}$ with two $10\text{ cm}\times 8\text{ cm}$ openings. What is the weight of the cement block to the nearest gram? (The density of cement is 1.7 gram / cm^3)

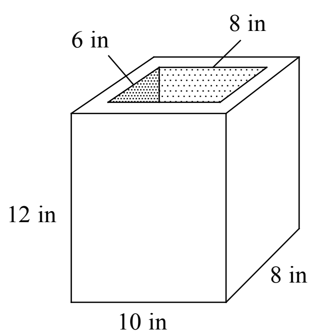
A) $5,040$

B) $6,048$

C) $7,560$

D) $8,568$

2



The figure above shows an aluminum block of $10\text{ in}\times 8\text{ in}\times 12\text{ in}$ with an $8\text{ in}\times 6\text{ in}\times 12\text{ in}$ opening. What is the weight of the aluminum block to the nearest pound? (The density of aluminum is 0.098 lb/in^3)

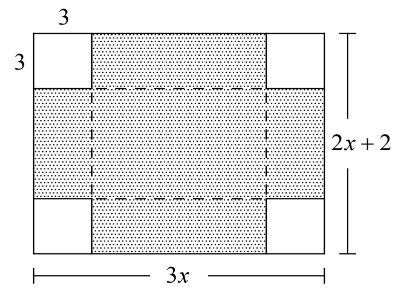
A) 32

B) 38

C) 42

D) 48

3



A manufacturing company produces cardboard boxes by cutting out square corners 3 inches (in) by 3 in. from rectangular pieces of cardboard $3x$ in. by $2x+2$ in. The cardboard is then folded along the dashed lines to form a box without a top. If the volume of the box is 162 in^3, what is the dimension of the original cardboard before cutting out its square corners?

A) $12\text{ in}\times 9\text{ in}$

B) $14\text{ in}\times 10\text{ in}$

C) $15\text{ in}\times 12\text{ in}$

D) $16\text{ in}\times 14\text{ in}$

4

An aquarium tank in the shape of a rectangular prism is 20 inches (in) long by 16 in wide by 12 in high. If 2,400 cubic inches of water is added into the empty tank, how far is the surface of the water from the top of the tank?

20-2. Cylinders and Spheres

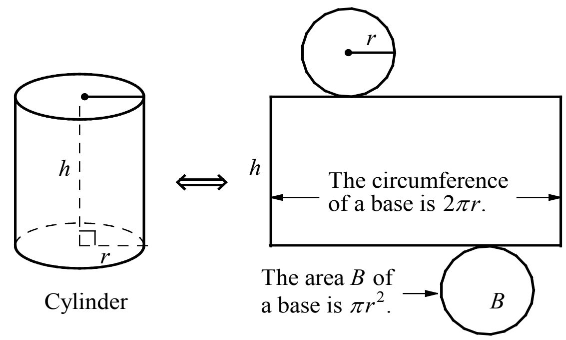
A **cylinder** is a solid that has two congruent parallel bases that are circles. In a right cylinder, the segment joining the centers of the bases is an altitude.

Theorems - Cylinders

The **lateral area** of a cylinder: $L.A. = 2\pi rh$

The total **surface area** of a cylinder: $T.A. = 2\pi rh + 2\pi r^2$

The **volume** of a cylinder: $V = \pi r^2 h$

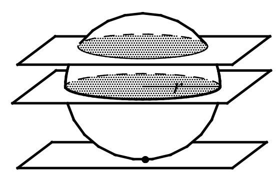


A **sphere** is the set of all points in space equidistant from a given point called the center. A plane can intersect a sphere at a point or in a circle. When a plane intersects a sphere so that it contains the center of the sphere, the intersection is called a **great circle**.

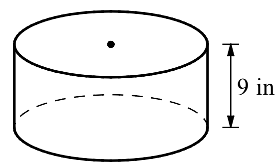
Theorems - Spheres

The **surface area** of a sphere: $S.A. = 4\pi r^2$

The **volume** of a sphere: $V = \frac{4}{3}\pi r^3$



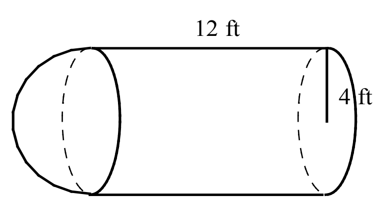
Example 1 □ The volume of the cylinder shown at the right is 225π cubic inches. What is the diameter of the base of the cylinder, in inches?



Solution □

$V = \pi r^2 h$	Volume of a cylinder
$225\pi = \pi r^2 (9)$	Substitution
$r^2 = \dfrac{225\pi}{9\pi} = 25$	Divide both sides by 9π
$r = \sqrt{25} = 5$	Square root both sides.
$\text{diameter} = 2r = 2 \cdot 5 = 10$	

Example 2 □ The figure at the right is a cylinder 12 feet high. It is capped by a hemisphere with a radius of 4 feet. What is the volume of the figure?



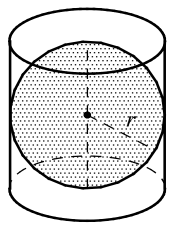
Solution □ Volume of cylinder $= \pi r^2 h = \pi(4)^2(12) = 192\pi$

Volume of hemisphere $= \frac{1}{2}(\frac{4}{3}\pi r^3) = \frac{1}{2} \cdot \frac{4}{3}\pi(4)^3 = \frac{128}{3}\pi$

Volume of the figure $= 192\pi + \frac{128}{3}\pi = \frac{704}{3}\pi$

Exercises - Cylinders and Spheres

1



In the figure above, a sphere is inscribed in a cylinder, so that the diameter of the sphere is the same as the diameter of the cylinder and the height of the cylinder. What is the value of $\dfrac{\text{Volume of the sphere}}{\text{Volume of the cylinder}}$?

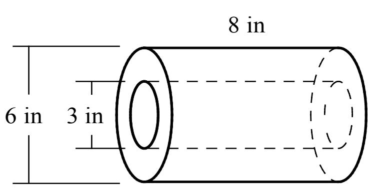
A) $\dfrac{1}{2}$

B) $\dfrac{2}{3}$

C) $\dfrac{7}{10}$

D) $\dfrac{3}{4}$

2



The figure above shows the mechanical part in the shape of a steel cylinder 8 inches high and 6 inches long in diameter. A hole with a diameter of 3 inches is drilled through the mechanical part. The density of steel is 490 lb/ft^3. What is the mass of the mechanical part, to the nearest pound? (1 foot = 12 inch)

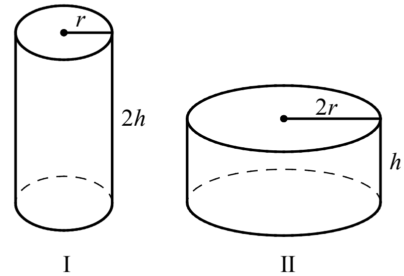
A) 36

B) 42

C) 48

D) 52

3



The figure above shows two cylinders. The height of cylinder I is twice the height of cylinder II and the radius of cylinder II is twice the radius of cylinder I. If the volume of cylinder I is $45\pi \text{ in}^3$, what is the volume of cylinder II in cubic inches?

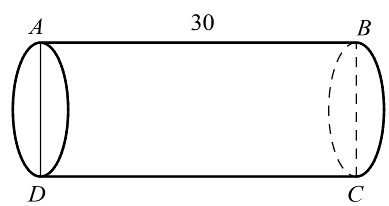
A) 22.5π

B) 45π

C) 67.5π

D) 90π

4



In the cylindrical tube shown above, the height of the tube is 30 and the circumference of the circular base is 32. If the tube is cut along $\overline{AB}$ and laid out flat to make a rectangle, what is the length of $\overline{AC}$ to the nearest whole number?

A) 24

B) 30

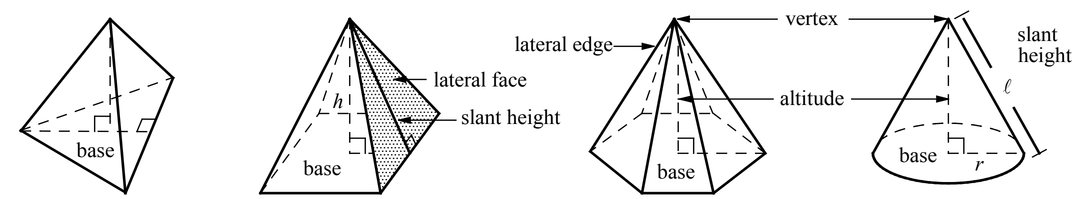
C) 34

D) 38

20-3. Pyramids and Cones

A **pyramid** is a polyhedron in which the base is a polygon and the **lateral faces** are triangles that meet at the **vertex**. The height of a lateral face is called the **slant height** of the pyramid, which is denoted by ℓ. A **regular pyramids** is a pyramid whose base is a regular polygon and whose lateral faces are congruent isosceles triangles. In SAT, you can assume that a pyramid is regular unless stated otherwise.

A **cone** is a solid that has a vertex and a circular base. The slant height ℓ is the distance from the vertex to a point on the edge of the base.



Triangular pyramid   Rectangular pyramid   Hexagonal pyramid   Cone

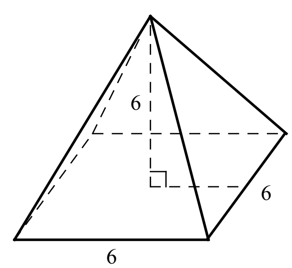
Theorems – Pyramids and Cones

The **lateral area** of a pyramid:	$L.A. = \frac{1}{2}P\ell$	The **lateral area** of a cone:	$L.A. = \pi r\ell$
The **surface area** of a pyramid:	$S.A. = L.A. + B$	The **surface area** of a cone:	$S.A. = L.A. + B$
The **volume** of a pyramid:	$V = \frac{1}{3}Bh$	The **volume** of a cone:	$V = \frac{1}{3}\pi r^2 h$

Example 1 □ A square pyramid is shown at the right.

a. Find the base area of the pyramid.

b. Find the volume of the pyramid.



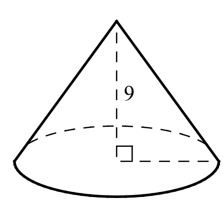
Solution □ a. Base area $= 6 \cdot 6 = 36$

b. $V = \frac{1}{3}Bh = \frac{1}{3}(36)(6) = 72$

Example 2 □ A cone with a height of 9 is shown at the right.

a. If the circumference of the base is 10π, what is the area of the base?

b. What is the volume of the cone?



Solution □ a. Circumference $= 2\pi r = 10\pi$

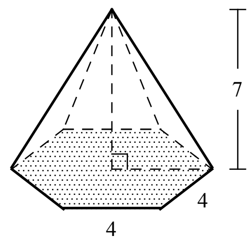
$\Rightarrow r = 5$

Base area $= \pi r^2 = \pi(5)^2 = 25\pi$

b. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5)^2(9) = 75\pi$

Exercises - Pyramids and Cones

1



The figure above shows a pyramid with regular hexagonal base. The length of each side of the hexagonal face is 4 units and the height of the pyramid is 7 units. What is the volume of the pyramid?

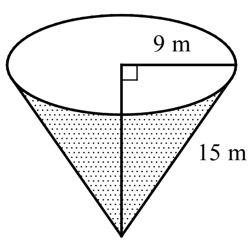
A) $35\sqrt{3}$

B) $56\sqrt{3}$

C) $84\sqrt{3}$

D) $168\sqrt{3}$

2



Water is pouring into a conical reservoir at the rate of 2.4 m^3 per minute. If the radius of the base of the conical reservoir is 9 meters (m) and the length of the lateral edge is 15 m, to the nearest minute, how long will it take to fill up the empty reservoir?

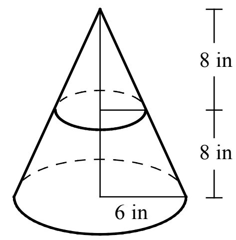
A) 212

B) 318

C) 424

D) 530

3



A plane parallel to the base of a cone divides the cone into two pieces, and removes the top part. The radius of the cone is 6 inches (in), the height of the cone is 16 in, and the distance from the base to the parallel plane is 8 in. What is the volume of the remaining bottom part, in cubic inches?

A) 56π

B) 84π

C) 126π

D) 168π

4

If the circumference of the base part of cone is 10 centimeters (cm) and the height of the cone is 8 cm, what is the volume of the cone, to the nearest cubic centimeter?

A) 18

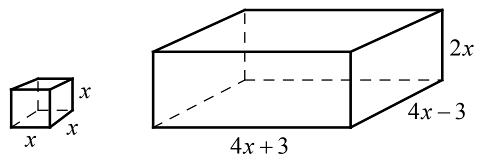
B) 21

C) 24

D) 32

Chapter 20 Practice Test

1



The figure above shows a cube and a rectangular prism. If the volume of the rectangular prism is 30 times the volume of the cube, what is the value of x?

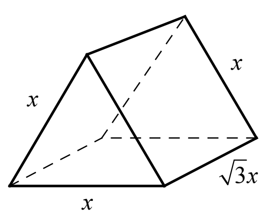
A) 1.5

B) 2

C) 2.5

D) 3

2



The figure above shows a triangular prism whose base is a equilateral triangle with side lengths x and height $\sqrt{3}x$. If the volume of the prism is $\frac{81}{4}$, what is the value of x?

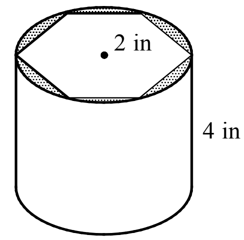
A) 3

B) 4

C) 5

D) 6

3



A regular hexagonal prism with edge lengths of 2 inches is created by cutting out a metal cylinder whose radius is 2 inches and height is 4 inches. What is the volume of the waste generated by creating the hexagonal prism from the cylinder, rounded to the nearest cubic inch?

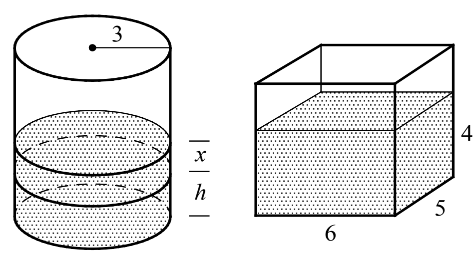
A) 7

B) 9

C) 11

D) 14

4



In the figure shown above, if all the water in the rectangular container is poured into the cylinder, the water level rises from h inches to $(h+x)$ inches. Which of the following is the best approximation of the value of x?

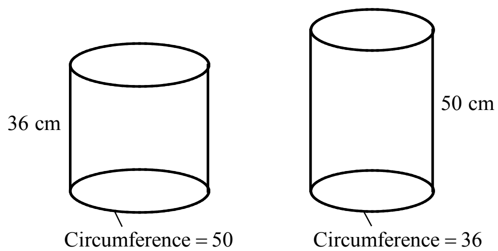
A) 3

B) 3.4

C) 3.8

D) 4.2

5



The figure above shows two cylinders that are rolled up from a poster 36 centimeter (cm) wide and 50 cm long without overlap. For cylinder I, the height is 36 cm and the circumference of the base is 50 cm. For cylinder II, the height is 50 cm and the circumference of the base is 36 cm. Which of the following is closest to the difference of volume between the two cylinders, in cubic centimeters?

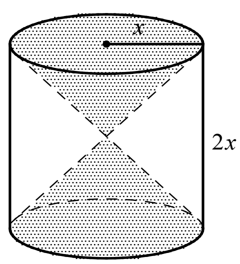
A) 1,600

B) 1,800

C) 2,000

D) 2,200

6



In the figure above, a double cone is inscribed in a cylinder whose radius is x and height is $2x$. What is the volume of the space inside the cylinder but outside the double cone, in terms of x?

A) $\frac{1}{2}\pi x^3$

B) $\frac{2}{3}\pi x^3$

C) $\frac{4}{3}\pi x^3$

D) $\frac{3}{2}\pi x^3$

7

The surface area of a cube is 54 square centimeters (cm^2). What is the volume of the cube in cubic centimeters?

8

A cone with a height of 10 cm and radius of 3 cm is 90 percent filled with shaved ice. What is the volume of the shaved ice, to the nearest cubic centimeter?

9

A square pyramid and a cube have equal volumes. The cube has an edge length of 4 inches and the pyramid has a base side length of 6 inches. What is the height of the pyramid in inches?

Answer Key

Section 20-1

1. D	2. B	3. C	4. 4.5

Section 20-2

1. B	2. C	3. D	4. C

Section 20-3

1. B	2. C	3. D	4. B

Chapter 20 Practice Test

1. D	2. A	3. B	4. D	5. C
6. C	7. 27	8. 85	9. $\frac{16}{3}$ or 5.33	

Answers and Explanations

Section 20-1

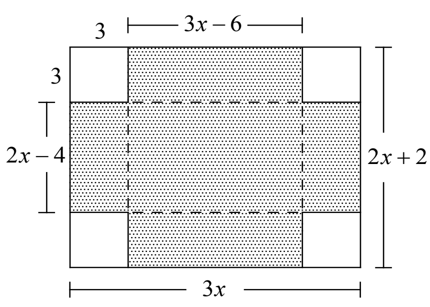
1. D

Volume of the block is $36\times 20\times 9 - 2(10\times 8\times 9)$, or $5{,}040 \text{ cm}^3$.
Weight of the cement block = density × volume $= 1.7 \text{ gram / cm}^3 \times 5{,}040 \text{ cm}^3 = 8{,}568 \text{ gram}$.

2. B

Volume of the aluminum block is $10\times 8\times 12 - 8\times 6\times 12$, or 384 in^3.
Weight of the aluminum block = density × volume $= 0.098 \text{ lb / in}^3 \times 384 \text{ in}^3 = 37.632 \text{ lb}$.

3. C



The dimension of the box is $(3x-6)\times(2x-4)\times 3$.
Since the volume of the box is given as 162 in^3, set up the following equation:

$$(3x-6)\times(2x-4)\times 3 = 162$$
$$(3x-6)\times(2x-4) = 54$$
$$6x^2 - 24x + 24 = 54$$
$$6x^2 - 24x - 30 = 0$$
$$6(x^2 - 4x - 5) = 0$$
$$6(x-5)(x+1) = 0$$
$$x = 5 \text{ or } x = -1$$

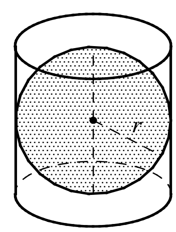
The length of the side is positive, so $x = 5$.
Therefore, the dimension of the original cardboard is $(3\times 5)\text{ in}\times(2\times 5+2)\text{ in}$, or $15\text{ in}\times 12\text{ in}$.

4. 4.5

Let h be the height of water when 2,400 cubic inches of water is added into the empty tank.
Then $20\times 16\times h = 2{,}400$. Solving for h yields
$h = \frac{2{,}400}{20\times 16} = 7.5 \text{ in}$.
Since the aquarium tank is 12 inches high, the surface of water will be $12 - 7.5$, or 4.5 inches, from the top of the tank.

Section 20-2

1. B



Volume of the sphere $= \frac{4}{3}\pi r^3$

Volume of the cylinder $= \pi r^2 \cdot \text{height}$
$= \pi r^2 \cdot 2r$
$= 2\pi r^3$

$$\frac{\text{Volume of the sphere}}{\text{Volume of the cylinder}} = \frac{\frac{4}{3}\pi r^3}{2\pi r^3} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

2. C

Volume of the mechanical part
$= \pi(3)^2 \cdot 8 - \pi(1.5)^2 \cdot 8 = 54\pi \text{ in}^3$

Since $1 \text{ ft} = 12 \text{ in}$, $1 \text{ ft}^3 = (12 \text{ in})^3 = 1{,}728 \text{ in}^3$.

Thus $1 \text{ in}^3 = \frac{1}{1{,}728}\text{ft}^3$.

Mass of the mechanical part $=$ density $\times$ volume

$= 490 \text{ lb / ft}^3 \times 54\pi \text{ in}^3 \cdot \dfrac{1 \text{ ft}^3}{1{,}728 \text{ in}^3} \approx 48.1 \text{ lb}$.

3. D

Volume of cylinder I $= \pi(r)^2 \cdot 2h = 2\pi r^2 h$

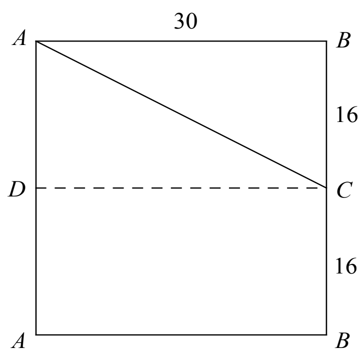
Volume of cylinder II $= \pi(2r)^2 \cdot h = 4\pi r^2 h$

$$\frac{\text{volume of cylinder I}}{\text{volume of cylinder II}} = \frac{2\pi r^2 h}{4\pi r^2 h} = \frac{1}{2}$$

Thus if the volume of cylinder I is $45\pi \text{ in}^3$, the volume of cylinder II is $90\pi \text{ in}^3$.

4. C

The figure below shows the rectangle, which was laid flat from a cut along $\overline{AB}$ on the cylinder.



$AC^2 = AB^2 + BC^2$ Pythagorean Theorem

$AC^2 = 30^2 + 16^2$ Substitution

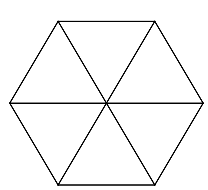
$= 900 + 256$

$= 1{,}156$

$AC = \sqrt{1{,}156} = 34$

Section 20-3

1. B



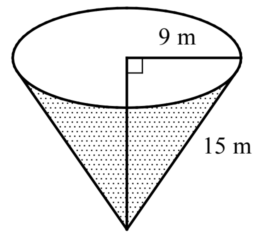
The regular hexagon consists of 6 equilateral triangles. So the area of the regular hexagon is the sum of the areas of 6 equilateral triangles. Since the area of the equilateral triangle with side length of a is $\dfrac{\sqrt{3}}{4}a^2$, the area of the equilateral triangle with side length of 4 is $\dfrac{\sqrt{3}}{4}(4)^2 = 4\sqrt{3}$.

The area of the hexagon is $6 \times 4\sqrt{3} = 24\sqrt{3}$.

Volume of the pyramid

$= \dfrac{1}{3}Bh = \dfrac{1}{3}(24\sqrt{3})(7) = 56\sqrt{3}$

2. C



Let $h =$ the height of the cone.

$9^2 + h^2 = 15^2$ Pythagorean Theorem

$h^2 = 15^2 - 9^2 = 144$

$h = \sqrt{144} = 12$

Volume of the cone $= \dfrac{1}{3}\pi r^2 h$

$= \dfrac{1}{3}\pi(9)^2(12) = 324\pi \text{ m}^3$

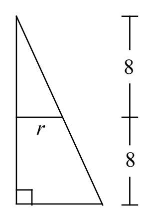
Number of minutes it takes to fill up the reservoir

$=$ total volume $\div$ rate of filling

$= 324\pi \text{ m}^3 \div 2.4 \text{ m}^3/\text{min}$

≈ 424.1 min

3. D



Let $r =$ the radius of the smaller cone.

Since the bases are parallel, the proportion $\dfrac{8}{r} = \dfrac{16}{6}$ can be used to find the radius of the smaller cone. $\dfrac{8}{r} = \dfrac{16}{6} \Rightarrow r = 3$

Volume of the remaining bottom part

$=$ volume of the cone $-$ volume of the top part

$= \dfrac{1}{3}\pi(6)^2(16) - \dfrac{1}{3}\pi(3)^2(8) = 168\pi$

4. B

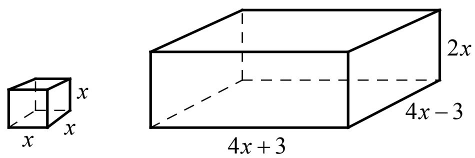
$$C = 2\pi r = 10 \Rightarrow r = \frac{10}{2\pi} = \frac{5}{\pi}$$

Volume of the cone

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{5}{\pi})^2(8) = \frac{200}{3\pi} \approx 21.2$$

Chapter 20 Practice Test

1. D



Volume of the rectangular prism

$= (4x+3)(4x-3)(2x) = (16x^2-9)(2x)$

Volume of the cube $= x^3$

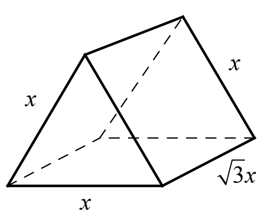
Since the volume of the rectangular prisms is 30 times the volume of the cube, the equation $(16x^2-9)(2x) = 30x^3$ can be used to find the value of x.

$(16x^2-9)(2x)-30x^3 = 0$	Make one side 0.
$2x\left[(16x^2-9)-15x^2\right] = 0$	GCF is $2x$.
$2x(x^2-9) = 0$	Simplify.
$2x(x+3)(x-3) = 0$	Factor.

$x = 0$, $x = -3$, and $x = 3$

Since the dimension has to be positive, $x = 3$ is the correct answer.

2. A



Area of the equilateral triangle with side length of x is $\frac{\sqrt{3}}{4}x^2$.

Volume of the triangular prism

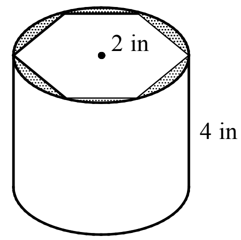
$$= B \cdot h = \frac{\sqrt{3}}{4}x^2 \cdot \sqrt{3}x = \frac{3}{4}x^3$$

Since the volume of the prism is given as $\frac{81}{4}$, the equation $\frac{3}{4}x^3 = \frac{81}{4}$ can be used to find the value of x.

$$\frac{3}{4}x^3 = \frac{81}{4} \Rightarrow 3x^3 = 81 \Rightarrow x^3 = 27$$

$$\Rightarrow x = \sqrt[3]{27} = 3$$

3. B



Area of the equilateral triangle with side length of $2 = \frac{\sqrt{3}}{4}(2)^2 = \sqrt{3}$.

Area of the regular hexagon $= 6\sqrt{3}$.

Volume of the hexagonal prism $= 6\sqrt{3} \cdot 4 = 24\sqrt{3}$

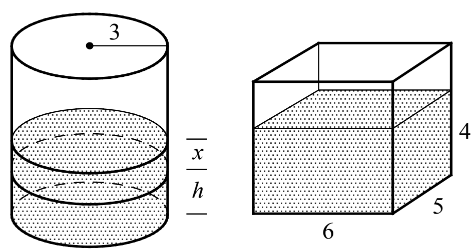
Volume of the cylinder $= \pi(2)^2 \cdot 4 = 16\pi$

The volume of the waste generated by creating the hexagonal prism from the cylinder can be found by subtracting the volume of the hexagonal prism from the volume of the cylinder.

$16\pi - 24\sqrt{3} \approx 8.69$

The volume of the waste is about 9 cubic inches.

4. D



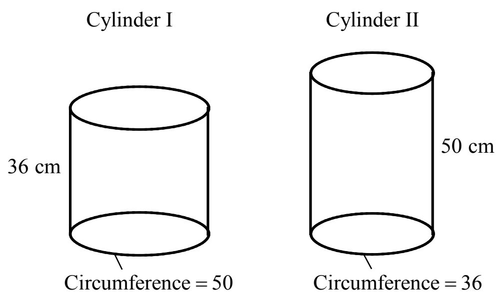
The volume of the cylinder with a radius of 3 and a height of x is $\pi(3)^2 x$, or $9\pi x$.

Volume of the water in the rectangular container is $6 \times 5 \times 4$, or 120.

To solve for x, let $9\pi x = 120$.

$$x = \frac{120}{9\pi} \approx 4.24$$

5. C



Let r_1 = the radius of cylinder I and
let r_2 = the radius of cylinder II.

$2\pi r_1 = 50 \Rightarrow r_1 = \frac{50}{2\pi} = \frac{25}{\pi}$

$2\pi r_2 = 36 \Rightarrow r_2 = \frac{36}{2\pi} = \frac{18}{\pi}$

Volume of cylinder I

$= \pi(r_1)^2 h = \pi(\frac{25}{\pi})^2(36) = \frac{22,500}{\pi}$

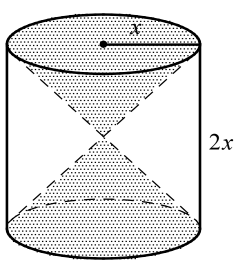
Volume of cylinder II

$= \pi(r_2)^2 h = \pi(\frac{18}{\pi})^2(50) = \frac{16,200}{\pi}$

The difference of volume between the two cylinders is $\frac{22,500}{\pi} - \frac{16,200}{\pi} \approx 2,005.3$.

Choice C is correct.

6. C



Volume of the space inside the cylinder but outside the double cone =
volume of the cylinder − volume of the two cones.

$\pi(x)^2(2x) - 2\left[\frac{1}{3}\pi(x)^2(x)\right]$

$= 2\pi x^3 - \frac{2}{3}\pi x^3 = \frac{4}{3}\pi x^3$

7. 27

Surface area of the cube $= 6s^2$

Since the surface area of the cube is given as 54 cm^2, $6s^2 = 54$.

$6s^2 = 54$ is simplified to $s^2 = 9$. Solving for s gives $s = 3$.

Volume of the cube $= s^3 = (3)^3 = 27$

8. 85

Volume of the cone $= \frac{1}{3}\pi(3)^2(10) = 30\pi$

Since the cone is 90 percent filled with shaved ice, the volume of the shaved ice is $30\pi \times 0.9$, or 27π cubic centimeters.

$27\pi \text{ cm}^3 \approx 84.8 \text{ cm}^3$

Therefore, to the nearest cubic centimeter, the volume of the shaved is 85 cm^3.

9. $\frac{16}{3}$ or 5.33

Let h = the height of the square pyramid.

Volume of the square pyramid

$= \frac{1}{3}Bh = \frac{1}{3}(6)^2 h = 12h$

Volume of the cube $= s^3 = (4)^3 = 64$

Since the square pyramid and the cube have equal volumes, $12h = 64$.

Solving for h gives $h = \frac{64}{12} = \frac{16}{3}$, or 5.33.